CENTRE DE RECHERCHE SUR LES TRANSPORTS
(C.R.T.)
CENTRE FOR RESEARCH ON TRANSPORTATION

UNIVERSITÉ DE MONTRÉAL
ÉCOLE DES HAUTES ÉTUDES COMMERCIALES
ÉCOLE POLYTECHNIQUE

CENTRE DE RECHERCHE SUR LES TRANSPORTS
UNIVERSITÉ DE MONTRÉAL
C.P. 6128, SUCCURSALE A,
MONTRÉAL
CANADA H3C 3J7

TÉLÉPHONE: (514) 343-7575
TÉLEX: (21) 05-561325
EMAIL: crt@crt.umontreal.ca
TÉLÉCOPIEUR (FAX): (514) 343-7121

92-02314/

CRT-716

Exact Solution of a Stochastic Location Problem by an Integer L-Shaped Algorithm

by

Gilbert Laporte[1]
François V. Louveaux[2]
Luc Van hamme[3]

[1] Centre de recherche sur les transports, Université de Montréal, C.P. 6128, succursale A, Mòntréal, Québec, Canada H3C 3J7

[2] Facultés Universitaires Notre-Dame de la Paix, Faculté des Sciences Économiques et Sociales, 8 Rempart de la Vierge, 5000 Namur, Belgium.

[3] Beyers Innovative Software, Michielssendreef 40/42, B-2130 Brasschaat, Belgium.

Centre de recherche sur les transports — Publication #716

September 1990
Revised January 1992

Dépôt légal – 1er trimestre 1992

Abstract

This paper considers a class of Capacitated Facility Location Problems in which customer demands are stochastic. The problem is formulated as a stochastic integer linear program, with first stage binary variables and second stage continuous variables. It is solved to optimality by means of an Integer L-Shaped Method. Computational results are reported for problems involving up to 40 customers and 10 potential facility locations.

Key words : Capacitated Facility Location, Stochastic Programming, Integer L-Shaped Method.

Résumé

On considère dans cet article une classe de problèmes de localisation d'installations avec capacités, dans lesquels les demandes des clients sont stochastiques. Le problème se formule comme un modèle de programmation stochastique en nombres entiers, avec variables binaires en première période et variables continues en seconde période. Il est ensuite résolu à l'aide d'une méthode de type "Integer L-Shaped". On présente des résultats numériques pour des problèmes comportant jusqu'à 40 clients et 10 sites de localisation potentiels.

Mots-clés : Problèmes de localisation d'installations avec capacités, Programmation stochastique, Méthode "Integer L-Shaped".

Introduction

One widely studied problem in location theory is the so-called *Uncapacitated Facility Location Problem* (UFLP). This problem consists of locating a number of facilities among a finite set of potential sites, in order to minimize a linear combination of fixed facility costs and of variable production and transportation costs to customers. The objective is to achieve the best tradeoff between fixed and variable costs: opening a larger number of facilities results in lower transportation costs at the expense of larger fixed costs, and conversely when fewer facilities are opened. The UFLP was originally studied by Kuehn and Hamburger (1963), Stollsteimer (1963), Manne (1964) and Balinski (1965) and is know to be NP-hard (Garey and Johnson 1979). The most efficient exact methods for this problem are dual based approaches (Bilde and Krarup 1977, Erlenkotter 1978, Körkel 1989).

Several UFLP extensions have been proposed. We are mostly interested here in *Capacitated Facility Location Problems* (CFLP) in which the production capacity of any facility is bounded above by a preset value. The CFLP has been studied by Akinc and Khumawala (1977), Geoffrion and McBride (1978), Guignard and Spielberg (1979) and more recently, Van Roy (1986). For an overview of these problems and their relationships to other location problems, see Hansen, Labbé, Peeters and Thisse (1987).

In recent years, a number of studies have addressed the stochastic dimension present in some facility location problems. One stream of research in this area concerns stochastic transportation models (Williams 1963, França and Luna 1982, Qi 1985, Wallace 1986). Another important problem is related to optimal location on networks with stochastic elements (Handler and Mirchandani 1979, Berman and LeBlanc 1984, Louveaux and Thisse 1985). Location models with

queueing elements have been studied by Brandeau and Chiu (1990) and Brandeau (1991). In addition, a number of papers have appeared on probabilistic location-covering models (ReVelle 1989, ReVelle and Hogan 1989).

The problem under consideration in this paper consists of optimally determining the location and size of facilities given that future customer demand is uncertain. Thus locating too few facilities, or facilities of inadequate size, may result in lost future sales. On the other hand, excessive investments will translate into unused capacity. As opposed to the deterministic case, demand is not known in advance and it is therefore more difficult to decide on the appropriate investment level. Our problem is thus a strategic decision problem under uncertainty and can be viewed as a two-stage stochastic integer program. In his paper, Louveaux (1986) has investigated the modeling of a class of related stochastic location problems and their relationship to stochastic p-median problems, while Psaraftis, Tharakan and Ceder (1986) provide an interesting application to the location and size of pollution control equipment. More recently, Louveaux and Peeters (1992) have proposed a heuristic dual-based procedure for the location and size of facilities when customer demands are described by a discrete random vector.

1. The model

Let the potential facility sites be denoted by $i = 1,...,n$ and customers by $j = 1,...,m$. In addition, define:

f_i	:	the fixed cost (per planning period) of opening a facility at site i;
f	=	$(f_1,...,f_n)$;
c_{ij}	:	the average cost (per planning period) of operating a transportation service between site i and customer j;
c	=	$(c_{11},...,c_{mn})$;

r_j : the non-negative unit net revenue of customer j;

R = $\max_j \{r_j\}$;

C_i : the maximum size of a facility located at site i;

ξ_j : a random variable representing the demand of customer j; it is assumed that for any $S \subseteq \{1,...,m\}$, the distribution function of $\sum_{j \in S} \xi_j$ can be computed;

ξ = $(\xi_1,...,\xi_m)$;

x_i : a binary variable equal to 1 if and only if a facility is located at site i;

x = $(x_1,...,x_n)$;

y_{ij} : a binary variable equal to 1 if and only if a transportation service is operated between i and j;

y = $(y_{11},...,y_{mn})$;

$w_{ij}(\xi)$: the quantity delivered from i to j in state of the world ξ;

$W(y,\xi)$: the revenue associated with y in state ξ;

$W(y)$ = $E_\xi W(y,\xi)$, where E_ξ denotes the mathematical expectation with respect to ξ.

As usual, it is assumed that all costs and revenues are defined for the same planning period. Note that the definition of c_{ij} assumes that the cost of servicing a link is related to the existence of that link only, and not to volume transported, such as in the case of several bus and train operations. The problem is then to

$$\text{minimize} \qquad z = \sum_{i=1}^{n} f_i x_i + \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij} y_{ij} - W(y), \tag{1}$$

subject to

$$y_{ij} \leq x_i \qquad (i = 1,...,n; j = 1,...,m) \tag{2}$$

$$\sum_{i=1}^{n} y_{ij} \leq 1 \qquad (j = 1,...,m) \tag{3}$$

$$x_i, y_{ij} \in \{0,1\} \qquad (i = 1,...,n; j = 1,...m), \tag{4}$$

where
$$W(y,\xi) = \max \sum_{i=1}^{n} \sum_{j=1}^{m} r_j w_{ij}(\xi), \tag{5}$$

subject to
$$0 \leq w_{ij}(\xi) \leq \xi_j y_{ij} \qquad (i = 1,...,n; j = 1,...,m) \tag{6}$$

$$\sum_{j=1}^{n} w_{ij}(\xi) \leq C_i \qquad (i = 1,...,n). \tag{7}$$

In this formulation, the three terms of the objective function denote the fixed cost associated with the facilities, the average cost of operating transportation services between facilities and customers, and the expected revenue derived from supplying customers, respectively. Constraints (2) impose that customers can only be served from open facilities, while constraits (3) require that each customer be served by only one facility. Constraints (5) compute for each state of the world ξ the maximum revenue under restrictions due to available demand (6) and available capacity (7). Finally, note that the second stage expected value function W is defined in terms of y only since in any meaningful first stage decision, $x_i = 1$ only if $y_{ij} = 1$ for at least one j.

2. Algorithm

The model just describe belongs to the class of $B/C/*$ stochastic problems where, using the notation developed by Laporte and Louveaux (1992), B means that the first stage decision variables are binary, C stands for continuous second stage variables, and $*$ indicates that the model applies to any type of random variable ξ (discrete, continuous or mixed). It can therefore be tackled by means of the *Integer L-Shaped Method* proposed by Laporte and Louveaux. Here follows a brief description of this algorithm as it applies to our problem.

First, define the *current problem* as

$$\text{minimize} \qquad z = \sum_{i=1}^{n} f_i x_i + \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} y_{ij} + \theta, \tag{8}$$

subject to (2), (3) and

$$\theta + E_k y \geq e_k \qquad (k = 1, ..., t) \tag{9}$$

$$\theta \geq -R \sum_{i=1}^{n} C_i x_i \tag{10}$$

$$\theta \geq L \tag{11}$$

$$0 \leq x_i \leq 1; 0 \leq y_{ij} \leq 1 \qquad (i = 1, ..., n; j = 1, ..., m), \tag{12}$$

where E_k is a vector with mn components, e_k is a scalar and k denotes the index of the so-called k^{th} *optimality cut* whose description will follow. Constraint (10) is a lower bound on the negative revenue θ, assuming the maximum revenue R is obtained for the full capacity of all facilities. Constraint (11) provides an alternative lower bound on θ, which will be derived later.

Step 0. Set $t := v := 0, \overline{z} := \infty$; the only pendent node corresponds to the initial current problem which is inserted in a list.

Step 1. Select a pendent node from the list; if none exists, stop.

Step 2. Set $v := v + 1$; solve the current problem. Let (x^v, y^v, θ^v) be an optimal solution.

Step 3. If $fx^v + cy^v + \theta^v > \overline{z}$, fathom the current problem and return to Step 1.

Step 4. Check for integrality restrictions. If one is violated, create two new nodes (corresponding to setting a fractional variable equal to 0 and 1); append these nodes to the list and return to Step 1.

Step 5. Compute $W(y^v)$ and $z^v := fx^v + cy^v - W(y^v)$. If $z^v < \overline{z}$, set $\overline{z} := z^v$ and store x^v, y^v.

Step 6. If $\theta^v \geq -W(y^v)$, fathom the current node and return to Step 1. Otherwise, impose one type (9) optimality cut, set $t := t + 1$ and return to Step 2. ■

We first provide details on the computation of L (constraint (11)). Two valid lower bounds on θ can be derived by alternatively relaxing constraints (6) or (7) in the definition of $W(y, \xi)$. In the first case, we obtain

$$W(y,\xi) \leq \sum_{i=1}^{n}\sum_{j=1}^{m} Rw_{ij}(\xi) = R\sum_{i=1}^{n}\left(\sum_{j=1}^{m} w_{ij}(\xi)\right) \leq R\sum_{i=1}^{n} C_i.$$

Taking expectations on both sides, we can write

$$-W(y) \geq -R\sum_{i=1}^{n} C_i. \tag{13}$$

Now relaxing (7) yields

$$W(y,\xi) \leq \sum_{i=1}^{n}\sum_{j=1}^{m} r_j\xi_j y_{ij} = \sum_{j=1}^{m} r_j\xi_j\left(\sum_{i=1}^{n} y_{ij}\right) \leq \sum_{j=1}^{m} r_j\xi_j.$$

Again, taking expected values gives

$$-W(y) \geq -\sum_{j=1}^{n} r_j\overline{\xi_j}, \tag{14}$$

where $\overline{\xi} = E_\xi(\xi_j)$. Combining (13) and (14) yields the desired lower bound:

$$L = \max\left(-R\sum_{i=1}^{n} C_i, -\sum_{j=1}^{m} r_j\overline{\xi}_j\right). \tag{15}$$

Since in a realistic context the total available capacity of open facilities will be close to the total expected demand, it is clear that the total capacity of all potential facilities will exceed the

total expected demand and therefore, the value of L should usually be equal to the second term of (15). Thus, (11) is usually not redundant with respect to (10).

We now explain the computation of $W(y^v)$ and the generation of constraints (9) for discrete and general random variables ξ.

Discrete case

If the random variables are discrete, constraints (9) can be obtained by using the optimal simplex multipliers, as is classically done in the continuous L-Shaped Method for two-stage stochastic programs with continuous second stage decision variables (Van Slyke and Wets 1969). These multipliers can easily be obtained from the computation of $W(y^v, \xi)$ for a given y^v and for every state ξ. This is done by applying a greedy procedure:

Step 1. Rank customers in non-increasing order of the r_j's.

Step 2. Assign to each open facility $i(x_i^v = 1)$ customer demands ξ_j in the same order as the r_j's, as long as $y_{ij}^v = 1$ and that C_i is not exceeded. Let γ_i^ξ be the largest index j such that $y_{ij}^v = 1$ and the demand of j cannot be fully satisfied by i; if no such j exists, let $\gamma_i^\xi = m + 1$, where $m + 1$ is a dummy customer with $r_{m+1} = 0$. ■

This procedure may of course leave customers with unsatisfied demands, or facilities with unsaturated capacities.

As in the continuous L-Shaped Method, we construct the optimality cut as

$$\theta + E_\xi \left(\sum_{i=1}^{n} \sum_{j=1}^{m} \xi_j u_{ij}^\xi y_{ij} \right) \geq -E_\xi \left(\sum_{i=1}^{n} C_i r_{\gamma_i^\xi} \right), \tag{16}$$

where

$$u_{ij}^{\xi} = \begin{cases} r_j - r_{\gamma_i^{\xi}} & \text{if } j < \gamma_i^{\xi} \\ 0 & \text{otherwise.} \end{cases}$$

General case (discrete, continuous or mixed)

Here, we base the generation of optimality cuts on the method proposed by Laporte and Louveaux (1992) — see the original paper for the various theoretical justifications. Define S^v as the set of pairs (i,j) for which $y_{ij}^v = 1$. Then constraints (9) can be written as

$$\theta + (W(y^v) + L)\left(\sum_{(i,j)\in S^v} y_{ij} - \sum_{(i,j)\notin S^v} y_{ij} \right) \geq (W(y^v) + L)(|S^v| - 1) + L,$$

where $|S^v|$ is the cardinality of S^v. An alternative expression of this optimality cut using fewer variables can be derived for our particular problem. First define

$J^+ = \left\{ j : \sum_{i=1}^{n} y_{ij} = 1 \right\}$: the set of customers served by a facility;

$J^- = \left\{ j : \sum_{i=1}^{n} y_{ij} = 0 \right\}$: the set of customers not served by any facility.

Also define for any $j \in J^+, i(j)$ as the facility serving j. Then the optimality cut can be written as

$$\theta + (W(y^v) + L)\left(\sum_{j\in J^+} y_{i(j),j} - \sum_{j\in J^-} \sum_{i=1}^{n} y_{ij} \right) \geq (W(y^v) + L)\left(|J^+| - 1\right) + L. \qquad (17)$$

This new optimality cut contains $|J^+|(n-1)$ fewer terms.

The general expression of $W(y^v)$ is given by

$$W(y^v) = E_{\xi} \sum_{i\in I(v)} \left(\sum_{j\in J(i,\xi,v)} r_j \xi_j + r_{v_i^{\xi}} \left(C_i - \sum_{j\in J(i,\xi,v)} \xi_j \right) \right), \qquad (18)$$

where $I(v) = \{i : x_i^v = 1\}$ and $J(i,\xi,v) = \left\{ j < v_i^{\xi} : y_{ij}^v = 1 \right\}$.

There exist a number of cases where (18) can readily be computed. For example, when $r_j = R$ for all j, the assignment of customers to open facilities i is only affected by C_i and does not depend on the relative customer unit revenues. In other words, (18) can be simply rewritten as a truncated expectation of the form

$$W(y^v) = R \sum_{i \in I(v)} E_\xi \left(\min \left(\sum_{j \in J(i,v)} \xi_j, C_i \right) \right), \tag{19}$$

where $J(i,v) = \left\{ j : y_{ij}^v = 1 \right\}$.

For several discrete random variables, the distribution of $\sum \xi_j$ is known and can be used to compute (19). A number of continuous cases are also computationally feasible. For example, if the $\xi_j's$ are independent normal variables with means μ_j and variances σ_j^2, then it can be shown that

$$W(y^v) = R \sum_{i \in I(v)} \left[C_i - (C_i - \overline{\mu}_i) F_i(C_i) - \left(\overline{\sigma}_i / \sqrt{2\pi} \right) \exp \left(-(C_i - \overline{\mu}_i)^2 / 2\overline{\sigma}_i^2 \right) \right], \tag{20}$$

where $\overline{\mu}_i = \sum_{j \in J(i,v)} \mu_j, \overline{\sigma}_i^2 = \sum_{j \in J(i,v)} \sigma_j^2$, and F_i is the cumulative function of the $N(\overline{\mu}_i, \overline{\sigma}_i^2)$ distribution. (It is implicitly assumed that the parameters $\overline{\mu}_i$ and $\overline{\sigma}_i^2$ are such that $P\left(\sum_{j \in J(i,v)} \xi_j < 0 \right)$ is negligible, which is reasonable in the case of a sum of demands).

3. Computational results

The algorithm described in the previous section was tested on a number of problems derived as follows. First, m customer locations were randomly generated according to a uniform distribution on $[0, 100]^2$. Customer demands ξ_j were assumed to be normally distributed with means $\mu_j \sim U[10, 50]$ and coefficients of variations $\sigma_j / \mu_j \sim U[0.5, 1.5]$. Three potential facility locations were randomly generated in the 10×10 square centered around each of the three

customers having the largest demands. The remaining $n-3$ facility locations were randomly generated in the square delimited by (20,20), (20,80), (80,20) and (80,80).

The problem parameters were chosen as follows. A control problem was first generated for $m = 20$, with all customers having the same (annual) revenue $R = 400$, all facilities having the same capacity $C = 300$ and an (annual) fixed cost $f = 10000$. The transportation service operating costs c_{ij} were set equal to the Euclidean distance between i and j, multiplied by a constant $D = 270$, equal to the approximate number of working days in a year.

Three different problems were attempted for the following values of the parameters, each modification being made from the control problem, one parameter at a time:

$$\begin{aligned} n \times m &= 5 \times 20, 10 \times 20; \\ R &= 300, 400, 500; \\ C &= 250, 300, 350; \\ f &= 5000, 10000, 20000. \end{aligned}$$

Similar problems were generated for $n \times m = 5 \times 10, 5 \times 40$ and 10×40, except that the parameters R, C and F were halved for $m = 10$, and doubled for $m = 40$. This choice was made to ensure a good balance between the three terms of (1), and a reasonable number of open facilities. In practice, the number of open facilities always turns out to be between 2 and 6, with a majority of problems having 3 or 4 open facilities when $n = 5$, and 3, 4 or 5 open facilities when $n = 10$. In total, 105 problems were generated.

All tests were performed on the Beyers & Partners VAX Station 3100 model 30, with a 3 MIPS processor. We used the OMP optimization software (De Decker *et al.* 1987) version 4, appropriately modified to encompass the branch and cut mechanism, the computation of (20) and the generation of the optimality cuts described in the previous sections. The maximum CPU time

allowed on any problem was 30 minutes.

Tests were carried out assuming every customer is assigned to a facility (i.e. (3) holds as an equality), so that (17) can be replaced with

$$\theta + (W(y^v) + L) \sum_{(i,j) \in S^v} y_{ij} \geq (W(y^v) + L)(m-1) + L.$$

Preliminary tests were made to compare constraints (2), $y_{ij} \leq x_i$, with the weaker form $\sum_{j=1}^{m} y_{ij} \leq m x_i$. As expected, the stronger form (2) is more efficient and was used in all tests. Experiments were also performed to compare two branching priorities: branching first on the opening decision variables x_i, or on the allocation variables y_{ij}. Differences were small and inconclusive, so that all further tests were executed by giving branching priority to the $x_i's$. Finally, experiments were made to compare choice rules for the first branching direction. "Upward first" branching seemed the most efficient.

Table 1 contains the main numerical results. These are averages on the problems which could be successfully solved within the prescribed time of 30 minutes. The meanings of the column headings are as follows:

SUCC	:	number of successful problems out of 3;
TIME	:	CPU time in seconds;
OPT-CUTS	:	total number of optimality cuts generated throughout the branch and cut process.

Two remarks are in order. First, CPU times include only the branch and cut part of the algorithm, excluding problem generation, data decoding, and the first LP. This time was usually of the order of a few seconds and never exceeded 15 seconds. Second, optimality cuts that were generated were maintained for all descendant nodes of the search tree, but removed when backtracking.

To better understand the influence of the constants R, C, f and D, a more detailed parametric

analysis was conducted on one of the 5×20 cases, the only size for which some problems were unsuccessful. Here, the tests were made on a single problem, the second most difficult of the three original 5×20 problems. Again, only one parameter is varied at a time. The results are given in Table 2. We only report the CPU time since the number of optimality cuts is related to it.

Table 1 Summary of computational results

$m = 10$	$n = 5$		
	SUCC	TIME	OPT-CUTS
CONTROL PROBLEM	3	2.72	23
R=300	3	1.72	14
R=500	3	4.07	34
f=2500	3	0.85	7
f=7500	3	5.28	37
C=125	3	11.99	152
C=175	3	0.89	7

$m = 20$	$n = 5$			$n = 10$		
	SUCC	TIME	OPT-CUTS	SUCC	TIME	OPT-CUTS
CONTROL PROBLEM	2	38.77	115	3	3.31	5
R=300	3	54.06	158	3	2.13	4
R=500	2	109.80	313	3	4.18	7
f=5000	2	55.22	244	3	1.86	1
f=15000	2	41.12	124	3	65.02	32
C=250	1	9.90	14	3	36.76	35
C=350	3	90.21	170	3	0.84	2

$m = 40$	$n = 5$			$n = 10$		
	SUCC	TIME	OPT-CUTS	SUCC	TIME	OPT-CUTS
CONTROL PROBLEM	3	1.25	4	3	12.19	7
R=300	3	0.96	3	3	6.44	5
R=500	3	1.48	5	3	14.77	9
f=10000	3	0.69	2	3	1.13	2
f=30000	3	12.07	23	3	16.80	7
C=500	3	22.20	56	3	488.10	327
C=700	3	0.75	2	3	0.77	19

Table 2 Parametric analysis for one 5 × 20 problem

R	300	350	400	500	600	700	800	
TIME	0.41	0.40	76.54	218.38	573.06	1306.53	>1800	
C	250	275	300	325	350			
TIME	>1800	935.57	76.54	10.66	1.30			
f	2500	5000	10000	15000	20000			
TIME	23.01	110.05	76.54	81.19	0.45			
D	200	220	240	260	270	280	290	300
TIME	1493.24	1076.99	746.86	372.59	76.54	6794	52.86	1.09

The following conclusions can be drawn from Tables 1 and 2. First, a large proportion of all attempted problems could be solved to optimality in less than one minute, and very often within a few seconds only. Second and more importantly, problem difficulty is not so much related to size as to the value of the various parameters. In particular, problem difficulty increases with R. This can be interpreted as follows: when revenue increases, so does the difference between θ and the correct second stage expected value W; these differences cannot be compensated by the first two terms of (1), so that for a given set of open facilities, several customer allocations must be examined. Similarly, when C or D decrease, a large number of allocations have to be considered, in the first case because allocating customers to the closest facility may induce a loss of revenue, in the second case since transportation cost plays a less important role. Finally, note that CPU times are not strongly affected by the fixed cost values.

4. Extension

A natural extension of the above problem is the following. Instead of assuming known facility capacities, we now consider these as decision variables. Let s_i be a variable equal to the size of facility i, $s = (s_i, ..., s_n)$, let g_i be its variable cost and $G = \min_i \{g_i\}$ The model is then

$$\text{minimize} \quad z = \sum_{i=1}^{n} (f_i x_i + g_i s_i) + \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} y_{ij} - W(s,y), \tag{21}$$

subject to (2), (3), (4) and

$$0 \leq s_i \leq C_i x_i \qquad (i = 1, ..., n), \tag{22}$$

where

$$W(s,y,\xi) = \max \sum_{i=1}^{n} \sum_{j=1}^{m} r_j w_{ij}(\xi), \tag{23}$$

subject to (6) and

$$\sum_{j=1}^{n} w_{ij}(\xi) \leq s_i \qquad (i = 1, ..., n). \tag{24}$$

Using again the Laporte and Louveaux taxonomy, this model belongs to the class of $M/C/*$ stochastic programs, where M stands for mixed integer first-stage decision variables. When the random variables are discrete, this model can again be solved by means of a natural extension of the integer L-Shaped Method, by substituting C_i by s_i in (16). The case of continuous random variables is more difficult, since no general algorithm exists for $M/C/C$ problems. We propose the following procedure valid for the particular case studied in Section 2, i.e., normal demands, $r_j = R$ for all j and $g_i \leq R$ for all i. (This last assumption is not restrictive since it is never profitable to open a facility whose unit variable costs are in excess of unit revenues).

Now define

$$W(y) = \min_{s} \left(\sum_{i=1}^{n} g_i s_i - W(s,y) \right), \tag{25}$$

subject to (22). The problem consisting of minimizing (21), subject to (2)-(4), (6) and (22)-(24) is then equivalent to minimizing (1), subject to (2)-(4), (6) and (23)-(25). This way of defining

$W(y)$ allows us to transform the current problem into: minimize (8), subject to (2), (3), (9), (11), (12) and

$$\theta \geq \sum_{i=1}^{n} (g_i - R)C_i x_i. \tag{26}$$

The current problem is now defined as previously, except for (26) which replaces (10). We have thus transformed the original $M/C/C$ problem into a $B/C/C$ problem. Note that this was done while maintaining s as a first-stage variable.

We can now apply the Integer L-Shaped Method to this problem, with new definitions for L and $W(y)$. In this problem, the value of L can be taken as

$$L = \max\left(\sum_{i=i}^{n} (g_i - R)C_i, (G - R)\sum_{j=1}^{n} \overline{\xi}_j\right). \tag{27}$$

In order to compute $W(s, y^v)$ at iteration v, it suffices to replace C_i by s_i in the right-hand side of (20). It follows that (26) subject to (22) is separable in i. For a given i. such that $x_i^v = 1$, we wish to minimize over s_i the function

$$\phi_i(s_i) = g_i s_i - Rs_i + R(s_i - \overline{\mu}_i)F_i(s_i) + R\left(\overline{\sigma}/\sqrt{2\pi}\right)\exp\left(-(s_i - \overline{\mu}_i)^2/2\overline{\sigma}_i^2\right), \tag{28}$$

subject to $0 \leq s_i \leq C_i$. The first-order derivative of ϕ_i reduces to

$$\phi_i'(s_i) \;=\; g_i \;-\; R[1 - F_i(s_i)],$$

so that the optimal value of s_i is equal to

$$s_i^* = \begin{cases} 0 & \text{if } F_i(0) > 1 - g_i/R \\ C_i & \text{if } F_i(C_i) < 1 - g_i/R \\ F_i^{-1}(1 - g_i/R) & \text{otherwise.} \end{cases} \tag{29}$$

The optimality cuts for this problem are again given by (17) with L as defined in (27) and $W(y^v) = \sum_{i \in I(v)} \phi_i(s_i^*)$.

The extent to which the suggested approach can be applied to other types of random variables remains to be investigated.

5. Conclusion

We have considered in this paper a difficult class of Capacitated Facility Location Problems with stochastic customer demands for which no exact algorithm had been previously developed. The problem belongs to the class of stochastic programs with binary first stage decision variables, continuous second stage decision variables, and discrete or continuous random variables. It was tackled by means of an Integer L-Shaped Method. Computational results confirm the efficiency and robustness of the proposed approach. Problems involving up to 40 customers and 10 potential facility locations were solved to optimality. However, difficulty is not so much related to problem size, as it is to the value of the parameters.

Acknowledgement

This work was partially supported by the Canadian Natural Sciences and Engineering Research Council (grant OGP0038682), and by the Collège Interuniversitaire d'Études Doctorales dans les Sciences du Management, Brussels.

References

1. U. Akinc and B.M. Khumawala, "An Efficient Branch and Bound Algorithm for the Capacitated Warehouse Location Problem", *Management Science*. 23, 585–594 (1977).
2. M.L. Balinski, "Integer Programming: Methods, Uses, Computation", *Management Science*. 12, 253–313 (1965).

3. O. Berman and B. LeBlanc, "Location-Relocation of N Mobile Facilities on a Stochastic Network", *Transportation Science*. 18, 315–330 (1984).

4. O. Bilde and J. Krarup, "Sharp Lower Bounds and Efficient Algorithms for the Simple Plant Location Problem", *Annals of Discrete Mathematics*. 1, 79–97 (1977).

5. M.L. Brandeau, "The Stochastic Queue Median Trajectory in a Plan with Generalized Distances", *Operations Research*. Forthcoming (1992).

6. M.L. Brandeau and S.S. Chiu, "A Unified Family of Single-Server Queueing – Location Models", *Operations Research*. 38, 1034–1044 (1990).

7. B. De Decker, F.V. Louveaux, C. Mortier, G. Schepens and A. Van Looveren, *Linear and Integer Programming with OMP*. Beyers & Partners, Brasschaat, Belgium, 1987.

8. D. Erlenkotter, "A Dual-Based Procedure for Uncapacitated Facility Location, *Operations Research*. 26, 992–1009 (1978).

9. P.M. França and H.P. Luna, "Solving Stochastic Transportation – Location Problems by Generalized Benders Decomposition". *Transportation Science*. 16, 113–126 (1982).

10. M.R. Garey and D.S. Johnson, *Computers and Intractibility. A Guide to the Theory of NP-Completeness*. W.H. Freeman, New York, 1979.

11. A.M. Geoffrion and R. McBride, "Lagrangian Relaxation Applied to Capacitated Facility Location Problems", *AIIE Transactions*. 10, 40–47 (1978).

12. M. Guignard and K. Spielberg, "A Direct Dual Method for the Mixed Plant Location Problem with Some Side Constraints", *Mathematical Programming*. 17, 198–228 (1979).

13. G.Y. Handler and P.B. Mirchandani, *Location on Networks. Theory and Algorithms*. MIT Press, Cambridge, Massachusetts, 1979.

14. P. Hansen, M. Labbé, D. Peeters and J.-F. Thisse, "Single Facility Location on Networks". In S. Martello, G. Laporte, M. Minoux, C. Ribeiro, eds., *Surveys in Combinatorial Optimization*. North-Holland, Amsterdam, 113–145, 1987.

15. M. Körkel, "Exact Solution of Large-Scale Simple Plant Location Problems", *European Journal of Operational Research*. 39, 157–173 (1989).

16. A.A. Kuehn and M.J. Hamburger, "A Heuristic Program for Locating Warehouses", *Management Science*. 9, 643–666 (1963).

17. G. Laporte and F.V. Louveaux, "The Integer L-Shaped Method for Stochastic Integer Programs", Publication CRT-788, Centre de recherche sur les transports, Montreal, 1992.

18. F.V. Louveaux, "Discrete Stochastic Location Models", *Annals of Operations Research.* 6, 23–24 (1986).

19. F.V. Louveaux and D. Peeters, "A Dual-Based Procedure for Stochastic Facility Location", *Operations Research.* Forthcoming (1992).

20. F.V. Louveaux and J.-F. Thisse, "Production and Location on a Network under Demand Uncertainty", *Operations Research Letters.* 4, 145–149 (1985).

21. A.S. Manne, "Plant Location under Economies-of-Scale – Decentralization and Computation", *Management Science.* 11, 213–235 (1964).

22. H.N. Psaraftis, G.G. Tharakan and A. Ceder, "Optimal Response to Oil Spills: The Strategic Decision Case", *Operations Research.* 34, 203–217 (1986).

23. L. Qi, "Forest Iteration Method for Stochastic Transportation Problem", *Mathematical Programming Study.* 25, 142–163 (1985).

24. C. ReVelle and K. Hogan, "The Maximum Reliability Location Problem and α-Reliable p-Center Problem: Derivatives of the Probabilistic Location Set Covering Problem", *Annals of Operations Research.* 18, 155–174 (1989).

25. J.F. Stollsteimer, "A Working Model for Plant Numbers and Locations", *Journal of Farm Economics.* 45, 631–645 (1963).

26. T.J. Van Roy, "A Cross Decomposition for Capacitated Facility Location", *Operations Research.* 34, 145–163 (1986).

27. R. Van Slyke and R.J-B Wets, "L-Shaped Linear Program with Applications to Control and Stochastic Programming", *SIAM Journal on Applied Mathematics.* 17, 638–663 (1969).

28. S.W. Wallace, "Decomposing the Requirement Space of a Transportation Problem into Polyhedral Cones", *Mathematical Programming Study.* 28, 29–47 (1986).

29. A.C. Williams, "A Stochastic Transportation Problem", *Operations Research.* 11, 759–770 (1963).